EIGENVALUE AND EIGENVECTOR SENSITIVITIES APPLIED TO POWER SYSTEM STEADY-STATE OPERATING POINT

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ABSTRACT

This paper proposes techniques used in finding a power system operating point that is both economically optimal and stable in the small-signal sense. To make the system small-signal stable, we should choose a power system operating point such that changes in system state variables due to small disturbances die out quickly. Thus, we need to know how changes in the power system operating point affect the linearized system's eigenvalues, i.e. the *eigenvalue sensitivities*. Methods to compute eigenvalue and eigenvector sensitivities are discussed.

1. INTRODUCTION

The optimal power flow (OPF) problem attempts to minimize some function of power system variables. Minimizing the cost of real power generation or minimizing real power losses in the system are examples of OPF objective functions.

The OPF is a constrained optimization problem. We have operating limits on voltage magnitudes and other system variables. Also, Kirchhoff's laws must be satisfied at the solution to the problem. Therefore, the solution to the problem is a power system steady-state operating point which may **not** be stable since the constraints historically included in the OPF problem are not stability constraints.

We begin the definition of stability by first noting that each generator has a set of nonlinear differential equations describing the synchronous machine, exciter, and any other control mechanisms. Each generator also has a set of algebraic equations which couple the generator state variables and the generator's steady-state operating point power injection into the system. Finally, there are the power system network equations, i.e. the Kirchhoff's law circuit equations that the steady-state operating point must satisfy.

At the steady-state operating point, we can linearize the set of differential equations, the algebraic equations, and the network equations. We then get a set of linear differential and algebraic equations which describe the power system dynamic response for small deviations from the operating point about which we linearized.

If the complex *eigenvalues* of the linearized system have negative real parts, then the power system can withstand small disturbances and is considered stable in the small-signal sense.

The linearized system eigenvalues depend on the steady-state operating point. Therefore, each eigenvalue is a function of power system variables. If we change one of these variables, the eigenvalue will experience some change. This is the eigenvalue sensitivity. Computing the eigenvalue sensitivity is of interest since we can include constraints on the eigenvalues in the OPF formulation. When minimizing our objective, we need to know how the eigenvalues will change when the optimization problem decision variables change.

In [1], eigenvalue sensitivities with respect to general system operating parameters were discussed. The sensitivities were used to determine how system loading levels affect stability. Reference [2] applies eigenvalue sensitivities to controller design. Our paper discusses some eigenvalue and eigenvector sensitivity computation techniques not presented in [1] or [2] and discusses their application to optimal power flow.

2. OPF WITH EIGENVALUE CONSTRAINTS

The problem we want to solve is

$$\min_{Y} f(Y) \tag{1}$$

subject to

$$h(Y) = 0, \tag{2}$$
$$g(Y) \leq 0. \tag{3}$$

where $f(Y)$ is the objective function, $h(Y)$ is the vector of equality constraints, and Y is the vector of decision variables including real and reactive power generation levels, bus voltages, and other variables. The equality constraints $h(Y)$ are the power balance equations

$$P_{Gi} - P_{Li} = P_i(V, \theta), \quad \forall i \in \mathcal{N} \tag{4}$$
$$Q_{Gi} - Q_{Li} = Q_i(V, \theta), \quad \forall i \in \mathcal{N} \tag{5}$$

where $\mathcal{N}$ is the set of all buses, V is the vector of bus voltage magnitudes, and θ is the vector of bus voltage phase angles. The vector of inequality constraints is $g(Y)$. These constraints include but are not limited to those on bus voltage magnitudes

$$V_i^{\min} \leq V_i \leq V_i^{\max}, \quad \forall i \in \mathcal{N}. \tag{6}$$

We propose the addition of small-signal stability inequality constraints by including limits on the real parts of the system state matrix eigenvalues. For the ith eigenvalue, this is written as

$$\mathbf{Re}(\lambda_i(Y)) \leq \lambda_i^{\max} < 0. \tag{7}$$

An alternate statement of stability constraints involves limiting both the real part of the eigenvalue and the *damping ratio*, which is defined as the ratio of the real part of the eigenvalue to the magnitude of the eigenvalue. Limiting the damping ratio forces any oscillations that occur due to small disturbances to die out quickly in a limited number of oscillation cycles.

3. PRIMAL-DUAL INTERIOR-POINT SOLUTION METHOD

A primal-dual interior-point algorithm [3] may be used to solve our power system optimization problem. We begin by adding a log barrier term to the Lagrangian to form the barrier function

$$f(Y) + \sum_{i=1}^{N_1} z_i\, h_i(Y) - \mu \sum_{i=1}^{N_2} \ln(-g_i(Y)). \tag{8}$$

To find a critical point, we solve the set of equations corresponding to the first-order necessary conditions for optimality

$$\nabla_Y f + J_h^T z + J_g^T w = 0, \tag{9}$$
$$h_i = 0, \qquad i = 1 \dots N_1 \tag{10}$$
$$g_i + s_i = 0, \qquad i = 1 \dots N_2 \tag{11}$$
$$w_i s_i = \mu, \qquad i = 1 \dots N_2 \tag{12}$$
$$w_i, s_i \geq 0, \qquad i = 1 \dots N_2. \tag{13}$$

where J_h and J_g are the equality and inequality constraint Jacobians, respectively. We choose to solve this system of equations iteratively using Newton's method, which involves writing a first-order Taylor series expansion

$$\begin{bmatrix} \nabla_Y^2 f + \sum_{i=1}^{N_1} z_i\, H_{hi} + \sum_{i=1}^{N_2} w_i H_{gi} & J_h^T & J_g^T & 0 \\ J_h & 0 & 0 & 0 \\ J_g & 0 & 0 & I \\ 0 & 0 & S & W \end{bmatrix} \begin{bmatrix} \Delta Y \\ \Delta z \\ \Delta w \\ \Delta s \end{bmatrix} = \begin{bmatrix} -f_Y - J_h^T z - J_g^T w \\ -h \\ -g - s \\ \mu\, e - S\, w \end{bmatrix} \tag{14}$$

to determine our update vector. Therefore, we must compute the gradient and Hessian of each equality and inequality constraint, and since some of our $g_i(Y)$ are constraints on eigenvalues, we must compute first- and second-order partial derivatives of eigenvalue functions. Expressions for these sensitivities will be reviewed in sections 6 through 9. But first, we need to define the system state matrix of which we are computing the eigenvalues.

4. LINEARIZED POWER SYSTEM MODEL

After linearizing the differential, algebraic, and network equations which describe the complete power system we are left with a set of linear differential equations and a set of algebraic equations [4] of the form

$$\Delta \dot{x} = A_1 \Delta x + B_1 \Delta I_g + B_2 \Delta V_g + E_1 \Delta U \tag{15}$$
$$0 = C_1 \Delta x + D_1 \Delta I_g + D_2 \Delta V_g \tag{16}$$
$$0 = C_2 \Delta x + D_3 \Delta I_g + D_4 \Delta V_g + D_5 \Delta V_l \tag{17}$$
$$0 = D_6 \Delta V_g + D_7 \Delta V_l \tag{18}$$

where 15 is the set of linearized generator differential equations, 16 is the set of generator algebraic equations, 17 is the set of network equations for generator buses, and 18 is the set of network equations for load buses. Variable descriptions are given in Table 1. The Δ preceding each variable in the equations refers to small deviations from the steady-state operating point about which we performed our linearization.

x	-	includes state variables of synchronous machines
I_g	-	includes direct-axis and quadrature-axis currents I_d, I_q of synchronous machines
V_g	-	includes voltage magnitudes and phase angles at generator buses
V_l	-	includes voltage magnitudes and phase angles at load buses

Table 1: Description of power system variables.

We would like to simplify these equations to obtain the form

$$\Delta \dot{x} = A_{\text{sys}} \Delta x + E_1 \Delta U. \tag{19}$$

The first step is to solve 16 for ΔI_g and substitute into 15 and 17. Assuming D_1 is invertible, the result in matrix notation is

$$\begin{bmatrix} \Delta \dot{x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 D_1^{-1} C_1 & B_2 - B_1 D_1^{-1} D_2 & 0 \\ C_2 - D_3 D_1^{-1} C_1 & D_4 - D_3 D_1^{-1} D_2 & D_5 \\ 0 & D_6 & D_7 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta V_g \\ \Delta V_l \end{bmatrix} + \begin{bmatrix} E_1 \\ 0 \\ 0 \end{bmatrix} \Delta U \tag{20}$$

Continuing in this fashion, we can arrive at the system state matrix A_{sys} as in 19. Each coefficient matrix A_1, A_2, B_2, etc. of 15-18 is a function of the OPF decision variables in Y. Since the A_{sys} matrix is computed from these coefficient matrices, then A_{sys} will change as Y changes. In the next sections, we see how changes in elements of Y affect the eigenvalues through the way they change A_{sys}.

5. ASSUMPTIONS

Suppose the system state matrix A_{sys} has complete sets of n distinct eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, right eigenvectors $\{\phi_1, \dots, \phi_n\}$, and left eigenvectors $\{\psi_1, \dots, \psi_n\}$, where right eigenvectors are column vectors and left eigenvectors are row vectors. The eigenvalues and eigenvectors satisfy

$$A_{\text{sys}}\, \phi_i = \lambda_i\, \phi_i \tag{21}$$

and

$$\psi_i\, A_{\text{sys}} = \lambda_i\, \psi_i\, . \tag{22}$$

We can form a matrix Φ containing the vectors $\{\phi_1, \dots, \phi_n\}$ as its columns, a matrix Ψ containing the vectors $\{\psi_1, \dots, \psi_n\}$ as its rows, and a matrix Λ which contains the complex scalars $\{\lambda_1, \dots, \lambda_n\}$ as its diagonal elements. Then in matrix notation, we have $A_{\text{sys}}\, \Phi = \Phi\, \Lambda$.

Suppose we have a right eigenvector ϕ_i corresponding to λ_i and a left eigenvector ψ_j corresponding to λ_j where $i \neq j$. Then the following holds true:

$$\psi_j\, \phi_i = 0. \tag{23}$$

Now suppose we have a right eigenvector ϕ_i and a left eigenvector ψ_i corresponding to the same eigenvalue λ_i. Then we have

$$\psi_i\, \phi_i = c_i \tag{24}$$

where $c_i \in \mathbb{C}$. Note that since scalar multiples of the eigenvectors are also eigenvectors, we can set $c_i = 1$ by normalizing 24, allowing us to simplify expressions that will appear later.

6. FIRST-ORDER EIGENVALUE SENSITIVITIES

Now we are ready to find $\frac{\partial \lambda_i}{\partial \epsilon}$ as shown in [2] where $\epsilon \in Y$. Note that A_{sys} is a function of the set of OPF decision variables Y. First, we take the partial derivative of both sides of 21 with respect to ϵ:

$$\frac{\partial A_{\text{sys}}}{\partial \epsilon}\,\phi_i + A_{\text{sys}}\,\frac{\partial \phi_i}{\partial \epsilon} = \frac{\partial \lambda_i}{\partial \epsilon}\,\phi_i + \lambda_i\,\frac{\partial \phi_i}{\partial \epsilon}\,. \tag{25}$$

Multiplying both sides of 25 on the left by ψ_i and simplifying results in

$$\frac{\partial \lambda_i}{\partial \epsilon} = \frac{\psi_i\,\frac{\partial A_{\text{sys}}}{\partial \epsilon}\,\phi_i}{\psi_i\,\phi_i}\,. \tag{26}$$

7. SECOND-ORDER EIGENVALUE SENSITIVITIES COMPUTED USING ALL EIGENVALUES & EIGENVECTORS

In this section, we review a method proposed by [2] and [5] to compute the second-order eigenvalue sensitivities. This method requires that we know all the eigenvalues and eigenvectors even though we may only wish to calculate the second-order sensitivity for one eigenvalue.

Suppose we are trying to compute $\frac{\partial^2 \lambda_i}{\partial \epsilon_2 \partial \epsilon_1}$ for some $\epsilon_1, \epsilon_2 \in Y$. The second-order sensitivity is then

$$\frac{\partial^2 \lambda_i}{\partial \epsilon_2 \partial \epsilon_1} = \frac{1}{\psi_i\,\phi_i}\left[\psi_i \frac{\partial^2 A_{\text{sys}}}{\partial \epsilon_2 \partial \epsilon_1}\,\phi_i + \psi_i \frac{\partial A_{\text{sys}}}{\partial \epsilon_1} \sum_{\substack{k=1\\k\neq i}}^{n} \beta_{ik}\phi_k + \psi_i \frac{\partial A_{\text{sys}}}{\partial \epsilon_2} \sum_{\substack{k=1\\k\neq i}}^{n} \alpha_{ik}\phi_k\right] \tag{27}$$

where

$$\alpha_{ij} = \frac{\psi_j\,\frac{\partial A_{\text{sys}}}{\partial \epsilon_1}\,\phi_i}{\psi_j\,\phi_j(\lambda_i - \lambda_j)} \tag{28}$$

$$\beta_{ij} = \frac{\psi_j\,\frac{\partial A_{\text{sys}}}{\partial \epsilon_2}\,\phi_i}{\psi_j\,\phi_j(\lambda_i - \lambda_j)}\,. \tag{29}$$

Again, this formula requires knowledge of all eigenvalues and eigenvectors which is a very large problem to solve when the system state matrix is very large.

8. SECOND-ORDER EIGENVALUE SENSITIVITIES COMPUTED USING A SINGLE EIGENVALUE AND EIGENVECTOR.

We would like to find the second-order eigenvalue sensitivities without having to know **all** the eigenvalues and eigenvectors. The second-order eigenvalue sensitivity can be written in terms of two first-order eigenvector sensitivities:

$$\frac{\partial^2 \lambda_i}{\partial \epsilon_2 \partial \epsilon_1} = \frac{1}{\psi_i\,\phi_i}\left[\psi_i \frac{\partial^2 A_{\text{sys}}}{\partial \epsilon_2 \partial \epsilon_1}\,\phi_i + \psi_i\left[\frac{\partial A_{\text{sys}}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1} I\right]\frac{\partial \phi_i}{\partial \epsilon_2} + \psi_i\left[\frac{\partial A_{\text{sys}}}{\partial \epsilon_2} - \frac{\partial \lambda_i}{\partial \epsilon_2} I\right]\frac{\partial \phi_i}{\partial \epsilon_1}\right]. \tag{30}$$

If we know the eigenvector sensitivities $\frac{\partial \phi_i}{\partial \epsilon_1}$ and $\frac{\partial \phi_i}{\partial \epsilon_2}$, then we can compute $\frac{\partial^2 \lambda_i}{\partial \epsilon_2 \partial \epsilon_1}$. In the next section, we review two ways to compute the eigenvector sensitivities.

9. FIRST ORDER EIGENVECTOR SENSITIVITIES

9.1. Method I

We start by rearranging 25 to give us

$$(A_{\text{sys}} - \lambda_i\,I)\,\frac{\partial \phi_i}{\partial \epsilon_1} = -\left(\frac{\partial A_{\text{sys}}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1}\,I\right)\,\phi_i\,. \tag{31}$$

By definition of the eigenvalue, $(A_{\text{sys}} - \lambda_i\,I)$ is singular, so we can't simply multiply both sides of 31 by $(A_{\text{sys}} - \lambda_i\,I)^{-1}$ to solve for the eigenvector sensitivity. However, it can be shown that the term on the right side of 31 is indeed in the range of $(A_{\text{sys}} - \lambda_i\,I)$ even though $(A_{\text{sys}} - \lambda_i\,I)$ is singular. We can use 31 and a normalization condition

$$\left\|\phi_i(\epsilon_1)\right\|^2 = \phi_i^*(\epsilon_1)\,\phi_i(\epsilon_2) = 1 \tag{32}$$

on the eigenvectors to solve for the real and imaginary parts of $\frac{\partial \phi_i}{\partial \epsilon_1}$. The system of equations to solve is

$$\begin{bmatrix} A_{\text{sys}} - \mathbf{Re}(\lambda_i\,I) & \mathbf{Im}(\lambda_i\,I) \\ -\mathbf{Im}(\lambda_i\,I) & A_{\text{sys}} - \mathbf{Re}(\lambda_i\,I) \\ \mathbf{Re}(\phi_i^*) & \mathbf{Im}(\phi_i^*) \end{bmatrix} \begin{bmatrix} \mathbf{Re}\left(\frac{\partial \phi_i}{\partial \epsilon_1}\right) \\ \mathbf{Im}\left(\frac{\partial \phi_i}{\partial \epsilon_1}\right) \end{bmatrix} = \begin{bmatrix} -\mathbf{Re}\left(\left(\frac{\partial A_{\text{sys}}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1}\,I\right)\,\phi_i\right) \\ -\mathbf{Im}\left(\left(\frac{\partial A_{\text{sys}}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1}\,I\right)\,\phi_i\right) \\ 0 \end{bmatrix}. \tag{33}$$

9.2. Method II

A second method for computing eigenvector sensitivities is proposed in [6]. The results of [6] will be given here with equations using our notation and assumptions. First, we can take the partial derivative with respect to ϵ_1 of both sides of 32 to obtain

$$\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1} + \left(\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1}\right)^* = 0. \tag{34}$$

This equation is true under the following condition:

$$\mathbf{Im}\left(\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1}\right) = 0. \tag{35}$$

In other words, if $\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1}$ is real, then 34 is satisfied. However, if 34 is satisfied, then it is not necessarily true that $\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1}$ is real. If $\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1}$ is real, then 34 becomes

$$2\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1} = 0 \tag{36}$$

i.e.

$$\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1} = 0. \tag{37}$$

Multiplying both sides of 37 on the left by ϕ_i, we have

$$\phi_i\phi_i^*\frac{\partial \phi_i}{\partial \epsilon_1} = 0. \tag{38}$$

Adding $\phi_i \phi_i^* \frac{\partial \phi_i}{\partial \epsilon_1}$ to the left side of 31, we have

$$\left(A_{\rm sys} - \lambda_i\, I + \phi_i \phi_i^*\right) \frac{\partial \phi_i}{\partial \epsilon_1} = -\left(\frac{\partial A_{\rm sys}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1}\, I\right)\, \phi_i\, . \quad (39)$$

Adding $\phi_i \phi_i^*$ to $(A_{\rm sys} - \lambda_i\, I)$ is a rank-1 modification of $(A_{\rm sys} - \lambda_i\, I)$. Since $\left(\frac{\partial A_{\rm sys}}{\partial \epsilon_1} - \frac{\partial \lambda_i}{\partial \epsilon_1}\, I\right)\, \phi_i$ is in the range of $(A_{\rm sys} - \lambda_i\, I + \phi_i \phi_i^*)$, then we need only solve the system of equations given by 39 to compute the eigenvector sensitivity. This method is different from the previous in the way the normalization condition is handled.

As emphasized earlier, both methods need only knowledge of the ith eigenvalue and left eigenvector to compute the eigenvector sensitivity. Once we know the eigenvector sensitivity, it can be plugged into 30 to obtain the second-order eigenvalue sensitivity.

10. CONCLUSIONS

This paper has discussed how the power system steady-state operating point affects system small-signal stability. This stability is reflected by the eigenvalues of the system state matrix, which are computed by linearizing the power system differential and algebraic equations.

The state matrix is a function of power system steady-state operating point variables. Changes in the steady-state operating point affect the eigenvalues through their effect on the state matrix as shown in the eigenvalue sensitivity expressions. Therefore, we can adjust the eigenvalues of our system by adjusting the steady-state operating point. Eigenvalue sensitivities tell us how to change the operating point to move eigenvalues to the desired location.

Eigenvalue sensitivity computational methods are based on knowledge of eigenvalues and eigenvectors. We emphasized that eigenvalue sensitivity techniques based on knowing a single eigenvalue and eigenvector were preferred in light of the Arnoldi algorithm [7], a method that can be used to find the set of *critical* eigenvalues.

11. REFERENCES

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